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LETTER TO THE EDITOR

Topological solitons in a sine–Gordon system with Kac–Baker long-range interactions

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Abstract. We derive an implicit form for topological solitons in a sine–Gordon system with a long-range interaction potential of the Kac–Baker type. The soliton width and energy are found to go to infinity as the long-range interaction is increased. The results recover those for the sine–Gordon system with nearest-neighbour interactions.

Recent studies on the effects of long-range interactions on dynamics and thermodynamics of anharmonic lattices have revealed interesting new phenomena [1,2 and references therein]. Among the various types of long-range interaction potential, a well studied example is the so-called Kac–Baker potential [3] in which the interaction between particles falls off exponentially as the separation increases. Defined as

$$v_{ij} = [C(1-r)/2r]r^{|i-j|} \tag{1}$$

it is encountered in systems undergoing phase transitions. The coefficient C is the elastic constant of the lattice. The parameter r defines the range of interaction with $r \in [0, 1[$ and can be seen as a measure of the ratio $V_{i,\ j+1}/V_{ij}$ of the elastic coupling coefficient between the ith particle, the jth and $(j+1)$th particles. The absolute difference $|i-j|$ measures the distance between the particles of sites i and j. Thus, when r increases, the range of interaction (the coupling coefficient V_{ij} between the particles on sites i and j) continuously increases. For a given r, V_{ij} decreases when j increases. Experimentally, one can relate the parameter r to the number of neighbouring interactions. This particular potential has been used to describe the thermodynamics of a one-dimensional Φ^4 system both in the continuum [4] and the discrete limits [2]. It has also been used to describe the dynamics of solitons in an anharmonic non-magnetic chain [5] and a magnetic Heisenberg chain [6].

Due to the mathematical complexity, the connection between the long-range interactions and the widely used sine–Gordon (SG) substrate potential has received limited investigation. By replacing the second partial derivative in the SG equation by an integral operator which contains both the short-range (local) and the long-range (non-local) interactions, Pokrovsky and Virosztek [7] analysed the problem of the finite exponent observed in the soliton density at zero temperature. Recently, Braun *et al* [8] have considered the Frenkel–Kontorova systems with the power laws and

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Kac–Baker potential. Taking the Kac–Baker interactions as perturbations, they have found a renormalization of kink parameters (i.e. the kink width increases while its non-linearity decreases with an increasing range of interaction). But a closed-form soliton solution has not yet been obtained. This paper gives a model solution for an implicit form of topological or kink solitons in the SG lattice with the Kac–Baker long-range interaction potential. The expressions of kink width L and energy E are derived. It is seen that they increase as the range of interaction increases. Our results recover those obtained in the short-range limit [9].

Let us consider the SG Hamiltonian with the Kac–Baker potential (1)

$$H = \tfrac{1}{2}m\sum_i \dot{u}_i^2 + \alpha\sum_i(1-\cos u_i) + \tfrac{1}{2}\sum_{j\neq i} V_{ij}(u_i - u_j)^2 \tag{2}$$

where u_i and $\dot{u}_i$ are the displacement and the velocity of the ith particle. m and α are respectively the mass of the particle and the amplitude of the SG substrate potential. The limit $r \to 0$ reduces to the nearest-neighbour problem and the limit $r \to 1$ defines the infinite-range problem. In the latter case, also known as the Van der Waals limit, the system may exhibit a continuous phase transition at a finite temperature [4].

The equation of motion for u_i which follows from (2) is

$$\ddot{u}_i + \alpha\sin u_i + 2Cu_i = L_i. \tag{3}$$

The auxiliary quantity L_i defined as

$$L_i = \frac{C(1-r)}{r}\sum_{j\neq i} r^{|i-j|}u_j \tag{4}$$

satisfies the following recursive relation

$$(r+r^{-1})L_i = L_{i+1} + L_{i-1} + [C(1-r)/r](u_{i+1} + u_{i-1} - 2ru_i). \tag{5}$$

Although the long-range interaction system with the Hamiltonian (2) is a purely discrete model, we can use the property of the Kac–Baker exponential interaction to obtain analytic results, or at least the first-order step solution of the discrete problem. Indeed, the recursive relation (5) contains only the nearest-neighbour terms (the local terms). Thus we can make use of the continuum approximation and write

$$u_i \to u(x,t) \qquad u_{i+1} + u_{i-1} \simeq 2u(x,t) + b^2u_{2x}$$

and

$$L_i \to L(x,t) \qquad L_{i+1} + L_{i-1} \simeq 2L(x,t) + b^2L_{2x}$$

where b is the lattice constant. The subscript $2x$ (or $2t$ hereafter) stands for the second partial (or time) derivative. Then substituting the above relations into equation (5) and replacing L by its continuous version of equation (3), we obtain

$$rmb^2u_{2x2t} + \alpha rb^2(\sin u)_{2x} + C(1+r)b^2u_{2x} - (1-r)^2(mu_{2t} + \alpha\sin u) = 0. \tag{6}$$

As expected, for $r = 0$, equation (6) reduces to the well known SG equation. An equation similar to (6) was derived recently by Roseneau [11] for a weakly non-linear 1D lattice with N neighbouring interactions by using a method which correctly preserves the essential features of the discrete system. But no special link was assumed between the coupling coefficients of different neighbouring interactions. Consequently, the coefficient of the u_{xxtt} term, as well as that of the non-linear interaction potential term, were given as sums over the N interacting particles. But, in our equation (6), the coefficient of u_{xxtt} depends on the parameter r which measures the range of interaction. This is due to the exponential form (link) of the elastic coupling coefficients between all the particles of the lattice.

To find the large-amplitude solutions (kinks and antikinks), we use the procedure of [4]. We neglect the fourth-order term in equation (6) in the spirit of the continuum approximation and also because this term vanishes for zero-velocity solitons and/or for $r = 0$. Assuming a soliton with a constant velocity V, we define the transformation

$$y = (x - Vt)/\xi$$

with

$$\xi^2 = \left[C(1+r)b^2 - mV^2(1-r^2)\right]/\alpha(1-r)^2 \tag{7}$$

and equation (7) takes the reduced form

$$u_{2y} + \sigma(\sin u)_{2y} = \sin u \tag{8}$$

with

$$\sigma = r\alpha b^2/[C(1+r)b^2 - mV^2(1-r)^2].$$

Assuming that σ is a positive parameter, the solution of equation (8) which corresponds to a displacement of particles from one minimum to the other of the substrate potential can be obtained by imposing the classical boundary conditions: u and $u_y \to 0$ when y goes to infinity. Then, equation (8) is integrated once to give

$$u_y^2 = \left(-2\cos u + \sigma\sin^2 u + 2\right)/(1 + \sigma\cos u)^2 \tag{9}$$

which, with the aid of tables of integrals [10], leads to

$$\begin{aligned}\mp(y - y_0) = {} & [(1+\sigma)^{1/2}/2]\log\Big(2\sin^2(u/2)/[\![2 + 2(1+2\sigma)\cos^2(u/2) \\ & + 4\{(1+\sigma)[1+\sigma\cos^2(u/2)]\cos^2(u/2)\}^{1/2}]\!]\Big) \\ & - \sigma^{1/2}\log\Big[\!\Big[4\{\sigma[1+\sigma\cos^2(u/2)\cos^2(u/2)\}^{1/2} \\ & - 2[1 + 2\sigma\cos^2(u/2)]\Big]\!\Big].\end{aligned} \tag{10}$$

Equation (10) defines a closed-form kink (positive sign) or antikink (negative sign) solution for the SG system with the Kac–Baker long-range interaction potential.

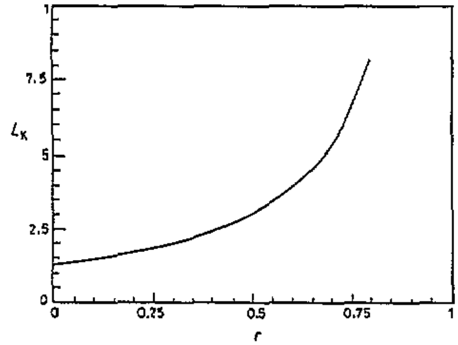

Figure 1. Kink width versus the long-range interaction parameter r.

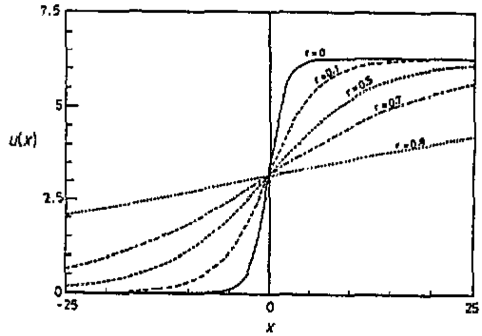

Figure 2. Kink variations versus the space variable x for different values of r.

As $r \to 0$, $\sigma \to 0$ and the implicit expression (10) reduces to the SG kink [9]

$$u = 4\tan^{-1}\exp\left[\mp(x - Vt - x_0)/d(1 - V^2/C_0^2)^{1/2}\right] \tag{11}$$

with $C_0^2 = Cb^2/m$ and $d^2 = Cb^2/\alpha$.

The parameter ξ gives a measure of the soliton width L_K. It increases with r (see figure 1) and the soliton slowly disappears. This is shown in figure 2 where u is plotted against x (with $V = 0$) for various values of the long-range interaction parameter r. Since the kink width L_K should be greater than a lattice spacing b, the soliton solution (10) requires that $C > \alpha$ and the parameter σ therefore varies from 0 to 1.

In the limit $r \to 1$, the soliton extension goes to infinity (diverges as $(1 - r)^{-1}$) and $u \to \pi$ for all x. This corresponds to the case in which all the particles sit at the top of the well of the SG substrate potential and have a maximum energy as shown below. The stability analysis of our soliton solution shows that there exists a bound state proportional to the spatial derivative u_x with zero frequency (the well known Golstone mode). The derivation of other eigenvalues and eigenfunctions from the linearized stability equation (obtained from equation (6) by substituting $u(x, t)$ by $u(x) + \psi(x, t)$ with $\psi(x, t) \ll u(x)$) has appeared to be an analytically difficult task since the soliton solution (10) has a complex implicit form. However the limiting case $r \to 0$ reduces to the well known stability equation of the SG model. Moreover, by looking for the asymptotic behaviour of the stability equation (x going to infinity), one can obtain the oscillations of particles around the bottom of the SG potential wells.

Using the auxiliary quantity L_i and equation (3) and going to the continuum limit, the potential energy of the system can be separated into three parts

$$E_p = E_1 + E_2 + E_3 \tag{12a}$$

with

$$E_1 = \alpha\int(1 - \cos u)\,\mathrm{d}x \tag{12b}$$

$$E_2 = -(m/2) \int u u_{2t} \, \mathrm{d}x \tag{12c}$$

$$E_3 = -(\alpha/2) \int u \sin u \, \mathrm{d}x \tag{12d}$$

where the integrals are taken along the infinite x axis.

Integrating equation (12c) by parts once, we obtain

$$E_2 = E_k = \frac{V^2}{2\xi} \int u_y^2 \, \mathrm{d}y \tag{12e}$$

where E_k is the soliton kinetic energy. We then substitute equation (9) and obtain after some cumbersome algebra the total energy E of the soliton in the form

$$E = \sigma\alpha\xi(1+\sigma)^{1/2} + \left(4mV^2/\xi\sigma^{1/2} + 2\alpha\xi/\sigma^{1/2} - 4\alpha\xi\sigma^{1/2}\right)$$
$$\times \log\left[(1+\sigma)^{1/2} + \sigma^{1/2}\right] + (4mV^2/\xi)\left[(1+\sigma)/\sigma(1-\sigma)\right]^{1/2} \tan^{-1}\left[\sigma/(1-\sigma)\right]^{1/2}. \tag{13}$$

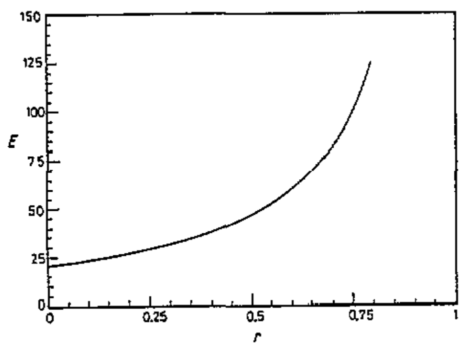

Figure 3. Soliton energy versus r.

An analysis of equation (13) shows that E goes to infinity as $r \to 1$. This state, as mentioned earlier, is energetically less favourable for the existence of the soliton since all the particles sit at the top of the well (an unstable position). In figure 3, we have given the variations of the soliton energy as a function of r. It is seen that E is an increasing function of the range of interaction. In the limit $r \to 0$, E reduces to

$$E = 8b(\alpha C)^{1/2}/(1 - V^2/C_0^2)^{1/2} \tag{14}$$

which corresponds to the well known kink energy of the SG model with nearest-neighbour interactions [9].

In summary, we have obtained an implicit expression for topological solitons in a 1D SG system interacting via the Kac–Baker potential. The width and the energy of solitons are found to go to infinity as the long-range interaction is increased, a behaviour qualitatively similar to that obtained in the Φ^4 model with the same potential [4]. In the short-range limit, the results reduce to that of the SG system with nearest-neighbour interactions. Thermodynamic properties of the model, both in the continuum and discrete limits, are currently under investigation and the results will be presented in the near future. Moreover, the effect of the long-range interaction on the central peak phenomena in the dynamic response function is a subject of considerable interest.

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